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Borsuk-Ulam theorem and formal group

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In this lecture I shall give a generalization of the classical Borsuk-Ulam theorem in connection with the formal group for a generalized cohomology theory.

This work was done in cooperation with Prof. Munkholm.

§1. Formal group law for multiplicative cohomology

We recall first some facts on multiplicative cohomology theory (see Dold [1]).

We fix once and for all a multiplicative ^{reduced} cohomology theory h defined on the category of finite CW complexes with base point. There is the ^{corresponding} multiplicative cohomology theory h defined on the category of finite CW pairs.

Let ξ be a real n -dimensional vector bundle over a finite CW complex B , and denote by $M(\xi)$ the Thom space for ξ . For each $b \in B$ let ξ_b denote the restriction of ξ over b . Then $\tilde{h}(M(\xi_b))$ is a free $h(pt)$ -module on one generator. ξ is said to be h -orientable if there exists $t(\xi) \in \tilde{h}^n(M(\xi))$ such that $t(\xi)|_{M(\xi_b)}$ is a generator of $\tilde{h}(M(\xi_b))$ for each $b \in B$. Such $t(\xi)$ is called an h -orientation or a Thom class of ξ . By an h -oriented vector

bundle we mean a vector bundle in which an h -orientation is given.

Let $D(\xi)$ (or $S(\xi)$) denote the total space of the disc bundle (or the sphere bundle) associated to ξ , and consider the homomorphism

$$\tilde{h}^n(M(\xi)) = h^n(D(\xi), S(\xi)) \xrightarrow{j^*} h^n(D(\xi)) \xrightarrow[p^*-1]{\cong} h^n(B),$$

where j is the inclusion and p is the projection. The image of $t(\xi)$ under this homomorphism is called the Euler class of the h -oriented bundle ξ , and is denoted by $e(\xi)$.

The following facts are easily proved:

(1.1) If there is a bundle map $f: \xi \rightarrow \xi'$ and ξ' is h -oriented, then ξ is h -oriented so that $f^*: h(B') \rightarrow h(B)$ preserves the Euler classes.

(1.2) If ξ_1 and ξ_2 are h -oriented, then the Whitney sum $\xi_1 + \xi_2$ is h -oriented so that $e(\xi_1 + \xi_2) = e(\xi_1)e(\xi_2)$.

(1.3) If ξ has a non-zero cross section, then $e(\xi) = 0$.

The classical Leray-Hirsch theorem on fibering can be generalized to the multiplicative theory h , and so we have Thom isomorphism

$$\tilde{\Phi}: h(B) \cong \tilde{h}(M(\xi))$$

given by $\tilde{\Phi}(\alpha) = \alpha \cdot t(\xi)$. As a consequence, the Gysin exact

sequence

$$\dots \rightarrow h^{i-1}(S(\xi)) \rightarrow h^{i-n}(B) \xrightarrow{\cdot e(\xi)} h^i(B) \xrightarrow{p^*} h^i(S(\xi)) \rightarrow \dots$$

holds.

A complex vector bundle ξ is called h -orientable if the real form ξ_R is h -orientable. Let γ_n denote the canonical complex line bundle over the complex n -dimensional projective space CP^n . Throughout this section the following will be assumed :

(1.4) For each n , γ_n is h -oriented so that the homomorphism $h(CP^{n+1}) \rightarrow h(CP^n)$ preserves the Euler classes.

It follows from this assumption that every complex line bundle ξ over a finite CW complex is h -oriented so that every homomorphism $f^* : h(B) \rightarrow h(B')$ induced by every bundle map $f : \xi \rightarrow \xi'$ preserves the Euler classes.

We can prove

(1.5) The algebra $h(CP^n)$ is a truncated polynomial algebra over $h(pt)$: $h(CP^n) = h(pt)[e(\gamma_n)] / (e(\gamma_n)^{n+1})$.

(1.6) Put $e(\gamma_m)_1 = p_1^* e(\gamma_m)$ and $e(\gamma_n)_2 = p_2^* e(\gamma_n)$.

for the projections $p_1 : CP^m \times CP^n \rightarrow CP^m$ and $p_2 : CP^m \times CP^n \rightarrow CP^n$. Then the isomorphism

$$h(CP^m \times CP^n) = h(pt)[e(\gamma_m)_1, e(\gamma_n)_2] / (e(\gamma_m)_1^{m+1}, e(\gamma_n)_2^{n+1})$$

holds.

For a CW complex X with finite skeleton, we define $h(X)$ as the inverse limit with respect to skeleton:

$$h(X) = \varprojlim h(X^n).$$

Then, for the infinite dimensional projective space CP^∞ , the following result is obtained from (1.5) and (1.6).

(1.7) $h(CP^\infty)$ and $h(CP^\infty \times CP^\infty)$ are rings of formal power series:

$$h(CP^\infty) = h[[x]], \quad h(CP^\infty \times CP^\infty) = h[[x_1, x_2]],$$

where x, x_1, x_2 are the elements defined by $e(\gamma_n), e(\gamma_n)_1, e(\gamma_n)_2$ respectively.

Let γ denote the canonical bundle over CP^∞ , and consider the external tensor product $\gamma \hat{\otimes} \gamma$ which is a complex line bundle over $CP^\infty \times CP^\infty$. Let $\mu : CP^\infty \times CP^\infty \rightarrow CP^\infty$ be a classifying map for $\gamma \hat{\otimes} \gamma$, ^{which is cellular} and put

$$\mu^*(x) = \sum_{i,j \geq 0} a_{ij} x_1^i x_2^j \quad (a_{ij} \in h^{2(1-i-j)}(pt))$$

for $\mu^* : h(CP^\infty) \rightarrow h(CP^\infty \times CP^\infty)$. Then we obtain easily

(1.8) For the tensor product $\xi_1 \otimes \xi_2$ of any complex line bundles ξ_1 and ξ_2 over a finite CW complex,

$$e(\xi_1 \otimes \xi_2) = \sum_{i,j \geq 0} a_{ij} e(\xi_1)^i e(\xi_2)^j$$

holds.

Consider now a power series $F(x,y)$ with coefficients in $h(pt)$, which is defined by

$$F(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j$$

with a_{ij} above. Then it follows that $F(x, y)$ is a formal group law over $h(pt)$, i.e. the identities

$$F(x, 0) = x = F(0, x), \quad F(x, y) = F(y, x),$$

$$F(x, F(y, z)) = F(F(x, y), z)$$

hold. For each integer $i \geq 1$, let $[i](x) \in h[[x]]$ denote the operation of "multiplication by i " for the formal group, i.e.

$$[1](x) = x, \quad [i](x) = F([i-1](x), x).$$

Since the formula in (1.8) is rewritten as

$$e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2)),$$

we have for the i -fold tensor product $\xi^i = \xi \otimes \cdots \otimes \xi$

$$e(\xi^1) = [1](e(\xi)).$$

Given a positive integer q , let G denote a cyclic group of order q . Define a G -action on the standard $(2n+1)$ -sphere $S^{2n+1} = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_1 |z_i|^2 = 1\}$ by

$$(z_0, \dots, z_n)g_0 = (z_0 \exp \frac{2\pi i}{q}, \dots, z_n \exp \frac{2\pi i}{q}),$$

where g_0 is the generator of G . This yields a principal G -bundle $\rho'_n : S^{2n+1} \rightarrow L^n(q)$ over the lens space $L^n(q)$.

Let L denote a 1-dimension complex G -module given by $c \cdot g_0 = c \exp 2\pi i/q$, and consider the associated complex line bundle $\rho_n = \rho'_n \times_G L$. For the canonical projection $\pi : L^n(q) \rightarrow \mathbb{C}P^n$ we have $\rho_n = \pi^*(\gamma_n)$, and hence $e(\rho_n)^{n+1} = 0$ holds.

Proposition 1. Let $P(x) \in h(pt)[[x]]$. Then the element $P(e(\rho_n))$ of $h(L^n(q))$ is zero if and only if $P(x)$ is in the ideal generated by x^{n+1} and $[q](x)$.

Proof. Consider the q -fold tensor product $\gamma_n^q = \gamma_n \otimes \dots \otimes \gamma_n$. Then it is easily checked that the total space $S(\gamma_n^q)$ of the sphere bundle associated to γ_n^q is homeomorphic with $L^n(q)$. Therefore we have the Gysin sequence

$$\dots \rightarrow h^{i-2}(\mathbb{C}P^n) \xrightarrow{\cdot e(\gamma_n^q)} h^i(\mathbb{C}P^n) \xrightarrow{\tau_*^*} h^i(L^n(q)) \rightarrow \dots$$

Since $e(\gamma_n^q) = [q](e(\gamma_n))$, the sequence and (1.5) derive the desired result (see [2]).

§ 2. The element $s^*(\theta)$

As in § 1, let G denote a cyclic group of order q .

We shall assume that q is odd.

For any space X , let XG denote the product of q copies of X . Writing its elements as $\sum_{g \in G} x_g g$, a G -action on XG is given by

$$\left(\sum_{g \in G} x_g g \right) \cdot h = \sum_{g \in G} x_{gh^{-1}} g \quad (h \in G).$$

We denote by ΔX the diagonal in XG .

Let Σ be a homotopy $(2n+1)$ -sphere (which is a differentiable manifold), and assume that there is given a free differentiable G -action on Σ . We denote by Σ_G the orbit space.

Let M be a differentiable manifold, and consider the diagonal action on $\Sigma \times MG$ whose orbit space is denoted by $\Sigma \times_G MG$. $\Sigma \times \Delta M$ is an invariant submanifold of the G -manifold $\Sigma \times MG$, and its orbit space is regarded as $\Sigma_G \times \Delta M$. We denote by ν the normal bundle of $\Sigma_G \times \Delta M$ in $\Sigma \times_G MG$.

This is a real $m(q-1)$ -dimensional vector bundle.

Choose a point $y_0 \in M$, and we shall identify Σ_G with a subspace $\Sigma_G \times y_0 G$ of $\Sigma_G \times \Delta M$.

Let $\lambda' : \Sigma \rightarrow \Sigma_G$ denote the principal G -bundle defined by the G -action on Σ , and consider the associated complex line bundle $\lambda = \lambda' \times_G L$.

We have the following (see [4])

Proposition 2. The normal bundle ν has a complex structure for which

$$\underline{i^*(\nu) = m(\lambda \oplus \lambda^2 \oplus \dots \oplus \lambda^{(q-1)/2})}$$

holds, where $i : \Sigma_G \rightarrow \Sigma_G \times \Delta M$ is the inclusion.

Proof. If $\nu_1 : N_1 \rightarrow \Delta M$ denote the normal G -vector bundle of ΔM in MG , then we have $\nu = 1 \times_G \nu_1 : \Sigma \times_G N_1 \rightarrow \Sigma_G \times \Delta M$. Therefore it suffices to prove that there exists an G -equivariant complex structure on ν_1 with the fiber over $y_0 G$ being $m(L \oplus \dots \oplus L^{(q-1)/2})$.

To prove this, let IG be defined by the exact sequence of real G -modules

$$0 \rightarrow \Delta R \rightarrow RG \rightarrow IG \rightarrow 0.$$

View this as a sequence of real G -vector bundles over a point, and identify ΔM with $M \times \text{pt} = M$ in the obvious way. Then we have the exact sequence

$$0 \rightarrow \tau M \hat{\otimes} \Delta \mathbb{R} \rightarrow \tau M \hat{\otimes} \mathbb{R} G \rightarrow \tau M \hat{\otimes} I G \rightarrow 0$$

of real G -vector bundle over M , where τM denotes the tangent bundle over M . Since $\tau(MG) = (\tau M)G$, an equivariant isomorphism

$$\beta : \tau(MG)|_{\Delta M} \rightarrow \tau M \hat{\otimes} \mathbb{R} G$$

can be given by

$$\beta \left(\sum_g v_g g \right) = \sum_g v_g \otimes g \quad (v_g \in \tau_y(M), y \in M).$$

Since $\sum_g v_g g$ is in $\tau(\Delta M)$ if and only if all v_g are equal, β maps $\tau(\Delta M)$ onto $\tau M \hat{\otimes} \Delta \mathbb{R}$. Thus it holds that $\nu_1 \cong \tau M \hat{\otimes} I G$ as real G -vector bundles. From elementary representation theory of groups, it follows that $I G$ is the real form of $L \oplus \dots \oplus L^{(q-1)/2}$. This gives ν_1 its complex structure, and we get

$$\begin{aligned} (\nu_1)_{y_0} &= \tau_{y_0} M \hat{\otimes} (L \oplus \dots \oplus L^{(q-1)/2}) \\ &= \mathbb{R}^m \hat{\otimes} (L \oplus \dots \oplus L^{(q-1)/2}) = m(L \oplus \dots \oplus L^{(q-1)/2}) \end{aligned}$$

as desired. This completes the proof.

As in 1, let h be a given multiplicative cohomology theory. In the following we shall assume the following conditions:

(2.1) every complex vector bundle of any dimension is h -orientable.

$$(2.2) \quad h^{\text{odd}}(\text{pt}) = 0.$$

Assuming that M is closed, consider the normal bundle ν . Then, by Proposition 2 and (2.1), we have a Thom class $t(\nu) \in \tilde{h}^{m(q-1)}(M(\nu))$ and the corresponding Euler class $e(\nu) \in h^{m(q-1)}(\sum_G \times M)$ such that

$$\begin{aligned} (2.3) \quad i^*e(\nu) &= e(m(\lambda \oplus \lambda^2 \oplus \dots \oplus \lambda^{(q-1)/2})) \\ &= \left(\prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \right)^m. \end{aligned}$$

As usual we shall regard the total space N of ν as a tubular neighborhood of $\sum_G \times M$ in $\sum_G^{\times} MG$. Then we can identify $\tilde{h}(M(\nu))$ with $h(\sum_G^{\times} MG, \sum_G^{\times} MG - N)$ canonically. Define

$$\dot{t} \in h^{m(q-1)}(\sum_G^{\times} MG)$$

to be the image of the Thom class $t(\nu)$ under the homomorphism

$q^* : h(\sum_G^{\times} MG, \sum_G^{\times} MG - N) \rightarrow h(\sum_G^{\times} MG)$ induced by the inclusion. We have immediately

(2.4) For the homomorphism $j^* : h(\sum_G^{\times} MG) \rightarrow h(\sum_G \times \Delta M)$ induced by the inclusion, $j^*(\theta) = e(\nu)$ holds.

Given a continuous map $f : \Sigma \rightarrow M$, define a continuous map $s : \sum_G \rightarrow \sum_G^{\times} MG$ by

$$s(xG) = (x, \sum_i f(xg_i^{-1})g_i)G.$$

For the projection $p : \sum_G^{\times} MG \rightarrow \sum_G$, $p \circ s$ is the identity.

Proposition 3. For the homomorphism $s^* : h(\sum_G^{\times} MG) \rightarrow h(\sum_G)$ and the homomorphism $i^* : h(\sum_G \times \Delta M) \rightarrow h(\sum_G)$,
we have

$$\underline{s^*(\theta) = i^*(e(\nu))}.$$

Proof. It is easily seen that there exist a continuous map $f_1 : \Sigma \rightarrow M$ and an open set V of Σ satisfying the following conditions: i) f is homotopic to f_1 , ii) V is homeomorphic to \mathbb{R}^{2n+1} , iii) $f_1(\Sigma - V) = y_0$, iv) $xg \notin V$ for any $g \neq 1$ and any $x \in V$. Define $s_1 : \sum_G \rightarrow \sum_G^{\times} MG$ from f_1 as in s , then s and s_1 are homotopic. Let $(Mg)_1$

denote the subspace of MG consisting of points with at most one coordinate $\neq y_0$. Then $(MG)_1$ is an invariant subspace of the G -space MG , and the orbit space $\Sigma_G^{\times} (MG)_1$ contains $s_1(\Sigma_G)$. Since $\Sigma - V$ is contractible, there exists a homotopy $\gamma_t : (\bar{V}, \partial V) \rightarrow (\Sigma, \Sigma - V)$ such that γ_0 is the inclusion and $\gamma_1(\partial V) = x_0$. Put $V_G = \pi(V)$ for the projection $\pi : \Sigma \rightarrow \Sigma_G$, and let \bar{V}, \bar{V}_G denote the closure of V, V_G respectively. Consider now the following diagram:

$$\begin{array}{ccccc}
 \Sigma_G & \xrightarrow{s_1} & \Sigma_G^{\times} (MG)_1 & & \\
 \downarrow j_2 & & \downarrow j_1 & & \\
 (\bar{V}_G, \partial V_G) & \xrightarrow{k} & (\Sigma_G, \Sigma_G - V_G) & \xrightarrow{s_1} & (\Sigma_G^{\times} (MG)_1, \Sigma_G \times y_0 G) \\
 \uparrow \pi & & & & \uparrow \\
 (\bar{V}, \partial V) & \xrightarrow{\gamma_1} & (\Sigma, x_0) & \xrightarrow{(1, f_1)} & (\Sigma \times M, \Sigma \times y_0),
 \end{array}$$

where j_1, j_2, k are the inclusions, and φ is given by

$$\varphi(x, y) = (x, y \cdot 1 + \sum_{g \neq 1} y_0 g)G.$$

Since a homotopy $\bar{\gamma}_t : (\bar{V}, \partial V) \rightarrow (\Sigma_G^{\times} (MG)_1, \Sigma_G \times y_0 G)$ of $s_1 \circ k \circ \pi$ to $\varphi \circ (1, f_1) \circ \gamma_1$ can be defined by

$$\bar{h}_t(v) = (h_t(v), f_1(h_t(v)) \cdot 1 + \sum_{g \neq 1} y_0 g)G \quad (v \in \bar{V}),$$

the above diagram is homotopy commutative. Therefore we

have $\pi^* \cdot k^* \cdot s_1^* = \psi_1^* \circ (1, f_1)^* \circ \rho^* : h(\sum_G^{\times} (MG)_1, \sum_G \times y_0 G) \rightarrow h(\bar{V}, \partial V)$.

We have

$$h^{m(q-1)}(\sum, x_0) = \tilde{h}^{m(q-1)}(S^{2n-1}) = h^{m(q-1)-(2n-1)}(\text{pt}) = 0$$

by (2.2), and k^* and π^* are isomorphisms. Therefore

$s_1^* : h^{m(q-1)}(\sum_G^{\times} (MG)_1, \sum_G \times y_0 G) \rightarrow h^{m(q-1)}(\sum_G, \sum_G - V_G)$ is trivial, and consequently

$$s_1^* \circ j_1^* : h^{m(q-1)}(\sum_G^{\times} (MG)_1, \sum_G \times y_0 G) \rightarrow h^{m(q-1)}(\sum_G)$$

is trivial.

Next consider the commutative diagram

$$\begin{array}{ccc}
 h(\sum_G^{\times} MG) & \xrightarrow{j^*} & h(\sum_G \times \Delta M) \\
 \downarrow s^* & \searrow i_2^* & \downarrow i^* \\
 & h(G(MG)_1) & \\
 \downarrow s_1^* & \uparrow p^* & \downarrow i_1^* \\
 h(\sum_G) & = & h(\sum_G) = h(\sum_G \times y_0 G)
 \end{array}$$

where i_1, i_2 are the inclusions. Putting $\theta' = p^* i_1^* i_2^*(\theta) - i_2^*(\theta)$ we have

$$s_1^*(\theta') = i^* j^*(\theta) - s^*(\theta) = i^*(e(\nu)) - s^*(\theta)$$

by (2.4), and $i_1^*(\theta') = 0$. Therefore θ' is in the image of $j_1^* : h^{m(q-1)}(\sum_G \times (MG)_1, \sum_G \times y_0 G) \rightarrow h^{m(q-1)}(\sum_G \times (MG)_1)$, and hence $s_1^*(\theta) = 0$ by the fact proved above. Thus we have proved $i^*(e(\)) = s^*(\)$.

§ 3. Generalization of Borsuk-Ulam theorem

Let Σ be as in § 2, and let $f : \Sigma \rightarrow M$ be a continuous map to a differentiable m -manifold. Put

$$A(f) = \{x \in \Sigma \mid f(x) = f(xg) \text{ for any } g \in G\}.$$

In this section we shall consider the covering dimension of $A(f)$.

For the image $A(f)_G = \pi(A(f))$, it follows from dimension theory that $\dim A(f) = \dim A(f)_G$.

Proposition 4. Assume that M is closed. Then $\dim A(f) < 2d$ implies $e(d\lambda) s^*(\theta) = 0$.

Proof. Since $\dim A(f)_G \leq 2d - 1$, it follows that $d\lambda$ has a non-zero cross section over $A(f)_G$ (see [3], Lemma 2). By standard facts on extension of cross section, this cross section extends to a non-zero cross section over the closure \bar{W} of some neighborhood W of $A(f)_G$ in Σ_G . Here we may assume that \bar{W} is a finite CW complex, and that $s(\Sigma_G - W) \subset \Sigma_G^{\times} MG - N$ by taking N small. We have then $e(d\lambda|_{\bar{W}}) = 0$, and so $e(d\lambda)$ is in the image of $\ell_1^* : h(\Sigma_G, \bar{W}) \rightarrow h(\Sigma_G)$ induced by the inclusion.

On the other hand, it follows from the commutative diagram

$$\begin{array}{ccc}
 h(\Sigma_G^{\times} MG, \Sigma_G^{\times} MG - N) & \xrightarrow{\ell^*} & h(\Sigma_G^{\times} MG) \\
 \downarrow s^* & & \downarrow s^* \\
 h(\Sigma_G, \Sigma_G - W) & \xrightarrow{\ell_2^*} & h(\Sigma_G)
 \end{array}$$

(ℓ, ℓ_2 : inclusions), that $s^*(\theta)$ is in the image of ℓ_2^* .

Therefore $e(d\lambda) s^*(\theta)$ is in the image of the homomorphism $h(\Sigma_G, \bar{W} \cup (\Sigma_G - W)) = h(\Sigma_G, \Sigma_G) \rightarrow h(\Sigma_G)$, and hence we have the desired result.

We shall now prove the main theorem.

Theorem 1. Let G be a cyclic group of odd order q ,

and Σ be a $(2n+1)$ -sphere on which a free differentiable G -action is given. Let M be a differentiable m -manifold. Assume that there exists a continuous map $f : \Sigma \rightarrow M$ with $\dim A(f) < 2d$. Then, for any multiplicative cohomology theory h , defined on the category of finite CW pairs and satisfying the conditions (2.1), (2.2),

$$\underline{x^d \left(\prod_{i=1}^{(q-1)/2} [i](x) \right)^m \in h[[x]]}$$

is contained in the ideal generated by x^{n+1} and $[q](x)$.

Proof. Recall that any differentiable m -manifold is regarded as an increasing union of compact differentiable m -manifold, and that any differentiable m -manifold with boundary is contained in a differentiable m -manifold without boundary. Since Σ is connected and compact, it follows from these facts that we may assume M to be closed without loss of generality.

Then, in virtue of (2.3), Prop. 3 and Prop. 4, we have

$$\begin{aligned} e(\lambda)^d \left(\prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \right)^m \\ = e(d\lambda) \cdot i^* e(\nu) = e(d\lambda) \cdot s^*(\theta) = 0. \end{aligned}$$

Since ρ'_n is a principal G -bundle whose base space is

$(2n+1)$ -dimensional CW complex, and since λ' is a $(2n+1)$ -universal principal G -bundle, there is a bundle map of f_n to λ . Hence the last equation implies

$$e(f_n)^d \left(\prod_{i=1}^{(q-1)/2} [i] (e(f_n)) \right)^m = 0$$

From this and Prop. 1 we have the desired result.

As typical examples of the multiplicative cohomology theory satisfying the conditions in Theorem, we have the classical integral cohomology theory $H^*(\ ; \mathbf{Z})$, the Grothendieck-Atiyah-Hirzebruch periodic cohomology theory $K^*(\)$ of K -theory, and the complex cobordism theory $U^*(\)$ obtained from the Milnor spectrum MU (see [5]).

As is well known, $H^i(\mathbb{P}^t; \mathbf{Z}) = \mathbf{Z}$ ($i = 0$), $= 0$ ($i \neq 0$) and the formal group law for $H^*(\ ; \mathbf{Z})$ is given by $F(x, y) = x + y$. Hence the conclusion in Theorem 1 for $h(\) = H^*(\ ; \mathbf{Z})$ is stated that

$$\left(\frac{q-1}{2} ! \right)^m x^{d+m(q-1)/2} \in \mathbf{Z}[x]$$

is contained in the ideal generated by x^{n+1} and qx .

From this we obtain the following result.

(3.1) If q is an odd prime, for any continuous map $f : \Sigma \rightarrow M$ we have $\dim A(f) \geq 2n - m(q-1)$.

Remark. The conclusion in (3.1) is strengthened to $\dim A(f) \geq 2n + 1 - m(q-1)$ (see [6], [7]).

For $K^*()$ it is known that $K^{\text{even}}(\text{pt}) = \mathbb{Z}$, $K^{\text{odd}}(\text{pt}) = 0$ and the formal group law is given by $F(x, y) = x + y + xy$. Therefore the conclusion in Theorem for $h() = K^*()$ is stated that

$$x^d \left(\prod_{i=1}^{(q-1)/2} ((x+1)^i - 1) \right)^m \in \mathbb{Z}[x]$$

is contained in the ideal generated by x^{n+1} and $(x+1)^q - 1$. Putting $y = x + 1$ this is restated that

$$(y-1)^d \left(\prod_{i=1}^{(q-1)/2} (y^i - 1) \right)^m \in \mathbb{Z}[y]$$

is contained in the ideal generated by $(y-1)^{n+1}$ and y^{q-1} . If q is an odd prime power p^a , it can be proved by making use of elementary algebraic number theory that the above statement is equivalent to

$$d \geq n + p^{a-1} - \frac{1}{2} \text{am}(p^a - p^{a-1})$$

(see [3]). Thus theorem 1 implies the following theorem containing (3.1) and being a generalization of a result in [3].

Theorem 2. If q is an odd prime power p^a , for any continuous map $f : \Sigma \rightarrow M$ we have

$$\underline{\dim A(f) \geq 2n + 2p^{a-1} - \dim(p^a - p^{a-1}) - 2.}$$

For $U^*()$ it is known that $U^*(pt)$ is a polynomial ring over \mathbb{Z} with one generator of degree $-2i$ for each positive integer i . However the formal group law for $U^*()$ is rather complicated (see e.g. [8]) and I have no method to derive numerical condition equivalent to the conclusion in Theorem 1. Since the cobordism theory is stronger than K-theory in general, it is expected that sharper result than Theorem 2 will be obtained from Theorem 1 for complex cobordism.

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